

MUMBAI UNIVERSITY

SEMESTER – 1

APPLIED MATHEMATICS SOLVED PAPER – DEC 18

N.B:- (1) Question no.1 is compulsory.

(2) Attempt any 3 questions from remaining five questions.

Q.1 (a) Show that $\sec h^{-1}(\sin \theta) = \log \cot \left(\frac{\theta}{2}\right)$. [3]

ANS: LHS = $\sec h^{-1}(\sin \theta)$

$$\text{Let } y = \sec h^{-1}(\sin \theta)$$

$$\sec hy = \sin \theta$$

$$\frac{1}{\sin \theta} = \frac{1}{\sec hy}$$

$$\cos hy = \operatorname{cosec} \theta$$

$$y = \cos h^{-1}(\operatorname{cosec} \theta)$$

$$\text{but } \cos h^{-1}x = \log |x + \sqrt{x^2 - 1}|$$

$$\therefore y = \log |\operatorname{cosec} \theta + \sqrt{\operatorname{cosec}^2 \theta - 1}|$$

$$\therefore y = \log |\operatorname{cosec} \theta + \cot \theta|$$

$$= \log \left| \frac{1}{\sin \theta} + \frac{\cos \theta}{\sin \theta} \right|$$

$$= \log \left| \frac{1 + \cos \theta}{\sin \theta} \right|$$

$$= \log \left| \frac{2 \cos^2 \frac{\theta}{2}}{2 \cos \frac{\theta}{2} \sin \frac{\theta}{2}} \right|$$

$$= \log \left| \frac{\cos \frac{\theta}{2}}{\sin \frac{\theta}{2}} \right|$$

$$= \log \cot \left(\frac{\theta}{2}\right).$$

$$= \text{RHS}$$

\therefore LHS = RHS

Hence proved

(b) Show that a matrix $A = \frac{1}{2} \begin{bmatrix} \sqrt{2} & -i\sqrt{2} & 0 \\ i\sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix}$ is unitary. [3]

ANS: Given $A = \frac{1}{2} \begin{bmatrix} \sqrt{2} & -i\sqrt{2} & 0 \\ i\sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix}$

To prove unitary, we have to prove $AA^\theta = I$

$$\therefore A^\theta = \frac{1}{2} \begin{bmatrix} \sqrt{2} & -i\sqrt{2} & 0 \\ i\sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

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$$\begin{aligned} \therefore \text{LHS} &= AA^\theta \\ &= \frac{1}{2} \begin{bmatrix} \sqrt{2} & -i\sqrt{2} & 0 \\ i\sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix} \frac{1}{2} \begin{bmatrix} \sqrt{2} & -i\sqrt{2} & 0 \\ i\sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix} \\ &= \frac{1}{4} \begin{bmatrix} 2+2+0 & -2i+2i+0 & 0+0+0 \\ 2i-2i+0 & 2+2+0 & 0+0+0 \\ 0+0+0 & 0+0+0 & 0+0+4 \end{bmatrix} \\ &= \frac{1}{4} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

LHS = I

= RHS

LHS = RHS

Hence proved.

(c) Evaluate $\lim_{x \rightarrow 0} \sin x \log x$. [3]

ANS: Let $L = \lim_{x \rightarrow 0} \sin x \log x$

$$L = \lim_{x \rightarrow 0} \frac{\log x}{\operatorname{cosec} x}$$

$$L = \lim_{x \rightarrow 0} \frac{\frac{1}{x}}{-\operatorname{cosec} x \cot x} \dots \dots \dots \text{(By L hospital)}$$

$$L = \lim_{x \rightarrow 0} \frac{-\sin x \tan x}{x}$$

$$L = \lim_{x \rightarrow 0} -\tan x$$

$$L = 0$$

(d) Find the n^{th} derivative of $y = e^{ax} \cos^2 x \sin x$. [3]

ANS: Given $y = e^{ax} \cos^2 x \sin x$

$$y = e^{ax} \left(\frac{1 + \cos 2x}{2} \right) \sin x$$

$$y = \frac{1}{2} (e^{ax} \sin x + e^{ax} \cos 2x \sin x)$$

$$y = \frac{1}{2} \left(e^{ax} \sin x + \frac{1}{2} e^{ax} (\sin 3x - \sin x) \right)$$

$$y = \frac{1}{2} \left(\frac{1}{2} e^{ax} \sin 3x + \frac{1}{2} e^{ax} \sin x \right)$$

Diff n times,

$$y_n = \frac{1}{2} \left(\frac{1}{2} e^{ax} (\sqrt{a^2 + 9})^n \sin \left(3x + n \tan^{-1} \frac{3}{a} \right) + \frac{1}{2} e^{ax} (\sqrt{a^2 + 1})^n \sin \left(x + n \tan^{-1} \frac{1}{a} \right) \right).$$

(e) If $x = r \cos \theta$ and $y = r \sin \theta$ prove that $JJ^{-1} = 1$. [4]

ANS: Given $x = r \cos \theta$ and $y = r \sin \theta$

i.e. $x, y \rightarrow f(r, \theta)$

$$\frac{\partial x}{\partial r} = \cos \theta \quad \frac{\partial x}{\partial \theta} = -r \sin \theta$$

$$\frac{\partial y}{\partial r} = \sin \theta \quad \frac{\partial y}{\partial \theta} = r \cos \theta$$

$$\therefore J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r (\cos^2 \theta + \sin^2 \theta) = r.$$

$$\therefore J = r \dots \dots \dots (1)$$

Now, to find values of r and θ

$$\therefore x^2 + y^2 = r^2 (\cos^2 \theta + \sin^2 \theta) = r^2$$

$$\therefore r = \sqrt{x^2 + y^2} \quad \text{and} \quad \frac{y}{x} = \frac{r \sin \theta}{r \cos \theta} = \tan \theta$$

$$\therefore \theta = \tan^{-1} \frac{y}{x}$$

$$\begin{aligned} \therefore J' = \frac{\partial(r,\theta)}{\partial(x,y)} &= \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix} \\ &= \frac{x^2}{(x^2+y^2)^{\frac{3}{2}}} + \frac{y^2}{(x^2+y^2)^{\frac{3}{2}}} \\ &= \frac{x^2+y^2}{(x^2+y^2)^{\frac{3}{2}}} \\ &= \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{r} \dots \dots \dots (2) \end{aligned}$$

From 1 and 2, we get

$$\text{Hence, } JJ' = r \cdot \frac{1}{r} = 1$$

Hence proved

(f) Using coding matrix $A = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$ encode the message THE CROW FLIES AT MIDNIGHT. [4]

ANS:

T = 20 H = 8 E = 5 C = 3 R = 18 O = 15 W = 23 F = 6 L = 12 I = 9 E = 5
S = 19 A = 1 T = 20 M = 13 I = 9 D = 4 N = 14 I = 9 G = 7 H = 8 T = 20

$$C = AB$$

$$B = \begin{bmatrix} 20 & 5 & 18 & 23 & 12 & 5 & 1 & 13 & 4 & 9 & 8 \\ 8 & 3 & 15 & 6 & 9 & 19 & 20 & 9 & 14 & 7 & 20 \end{bmatrix}$$

$$\therefore C = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 20 & 5 & 18 & 23 & 12 & 5 & 1 & 13 & 4 & 9 & 8 \\ 8 & 3 & 15 & 6 & 9 & 19 & 20 & 9 & 14 & 7 & 20 \end{bmatrix}$$

$$C = \begin{bmatrix} 48 & 13 & 51 & 52 & 33 & 29 & 22 & 35 & 22 & 25 & 36 \\ 68 & 18 & 69 & 75 & 45 & 34 & 23 & 48 & 26 & 34 & 44 \end{bmatrix}$$

Q.2] (a) Find all values of $(1 + i)^{\frac{1}{3}}$ and show that their continued product is $(1 + i)$. [6]

ANS: Let $Z = (1 + i)^{\frac{1}{3}}$

$$Z = [\sqrt{2} (\cos \frac{\pi}{4} + i \sin \frac{\pi}{4})]^{\frac{1}{3}}$$

$$Z = (\sqrt{2})^{\frac{1}{3}} \cdot [\cos(2k\pi + \frac{\pi}{4}) + i \sin(2k\pi + \frac{\pi}{4})]^{\frac{1}{3}}$$

$$Z = 2^{\frac{1}{6}} [\cos(\frac{8k\pi + \pi}{12}) + i \sin(\frac{8k\pi + \pi}{12})]$$

Putting $k = 0, 1, 2$.

$$Z_0 = 2^{\frac{1}{6}} \cdot e^{\frac{i\pi}{12}}$$

$$Z_1 = 2^{\frac{1}{6}} \cdot e^{\frac{9i\pi}{12}}$$

$$Z_2 = 2^{\frac{1}{6}} \cdot e^{\frac{17i\pi}{12}}$$

$$\therefore Z_0 Z_1 Z_2 = 2^{\frac{3}{6}} \cdot e^{\frac{27i\pi}{12}}$$

$$= 2^{\frac{1}{2}} \cdot e^{\frac{9i\pi}{4}}$$

$$= \sqrt{2} (\cos \frac{9\pi}{4} + i \sin \frac{9\pi}{4})$$

$$= \sqrt{2} (\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}})$$

$$= (1 + i).$$

(b) Find the non-singular matrices P & Q such that PAQ is in normal form

where $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$. [6]

ANS. Given matrix is $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$

The order of matrix is 3×4

$\therefore A = I_3 A I_4$.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Operate $R_2 - 2R_1$; $R_3 - 3R_1$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -5 \\ 0 & -6 & -4 & -22 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Operate $C_2-2C_1; C_3-3C_1; C_4-4C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & -2 & -5 \\ 0 & -6 & -4 & -22 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 & -4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Operate $\frac{C_2}{-3}; \frac{C_3}{-2}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -5 \\ 0 & 2 & 2 & -22 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & \frac{2}{3} & \frac{3}{2} & -4 \\ 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Operate R_3-2R_2

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & -12 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & \frac{2}{3} & \frac{3}{2} & -4 \\ 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Operate C_{34}

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -12 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & \frac{2}{3} & -\frac{2}{3} & \frac{5}{6} \\ 0 & -\frac{1}{3} & -\frac{5}{3} & \frac{1}{3} \\ 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Operate $\frac{R_3}{-12}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -\frac{1}{12} & \frac{1}{6} & -\frac{1}{12} \end{bmatrix} A \begin{bmatrix} 1 & \frac{2}{3} & -\frac{2}{3} & \frac{5}{6} \\ 0 & -\frac{1}{3} & -\frac{5}{3} & \frac{1}{3} \\ 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$[3, 0] = \text{PAQ}$ ie PAQ is in normal form,

Where,
$$P = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ \frac{-1}{12} & \frac{1}{6} & \frac{-1}{12} \end{bmatrix}$$
 and
$$Q = \begin{bmatrix} 1 & \frac{2}{3} & \frac{-2}{3} & \frac{5}{6} \\ 0 & \frac{-1}{3} & \frac{-5}{3} & \frac{1}{3} \\ 0 & 0 & 0 & \frac{-1}{2} \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

(c) Find maximum and minimum values of $x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$. [8]

ANS: Given $f(x) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$... (1)

STEP 1] for maxima, minima, $\frac{\partial f}{\partial x} = 0$; $\frac{\partial f}{\partial y} = 0$

$$3x^2 + 3y^2 - 30x + 72 = 0 \quad \text{and} \quad 6xy - 30y = 0$$

$$\therefore y(6x - 30) = 0$$

$$y = 0, x = 5$$

For $x = 5$; From Equation $3x^2 + 3y^2 - 30x + 72 = 0$, we get $y^2 - 1 = 0$

$$Y = \pm 1$$

Hence (4,0), (6,0), (5,1), (5,-1) are the stationary points.

STEP 2] Now, $r = \frac{\partial^2 f}{\partial x^2} = 6x - 30$;

$$S = \frac{\partial^2 f}{\partial x \partial y} = 6y$$

$$t = \frac{\partial^2 f}{\partial y^2} = 6x - 30$$

STEP 3] for (x, y) = (4, 0), $r = -6$, $s = 0$, $t = -6$;

$$rt - s^2 = (-6)(-6) - 0 = 36 > 0 \quad \text{and} \quad r < 0.$$

This shows that the function is maximum at (4, 0)

\therefore From Equation (1)

$$F_{\max} = f(4, 0) = 4^3 + 0 - 15(4^2) + 0 + 72(4) = 64 - 240 + 288$$

$$F_{\max} = 112$$

STEP 4] For (x, y) = (6, 0)

$$r = 6, s = 0, t = 6$$

$$rt - s^2 = 36 \text{ but } r = 6 > 0$$

This shows that function is minimum at (6, 0).

From Equation (1),

$$F_{\min} = f(6, 0) = 6^3 + 0 - 15(6)^2 + 0 + 72(6) = 108.$$

STEP 5] For (x, y) = (5, 1)

$$r = 0, s = 6, t = 0$$

$$(rt - s^2) < 0$$

This shows that at (5, 1) and (5, -1) function is **neither maxima nor minima.**

These points are **saddle points.**

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Q.3] (a) If $u=e^{xyz} f\left(\frac{xy}{z}\right)$ prove that $x\frac{\partial u}{\partial x} + z\frac{\partial u}{\partial z}=2xyzu$ and $y\frac{\partial u}{\partial x} + z\frac{\partial u}{\partial z}=2xyzu$ and hence show that $x\frac{\partial^2 u}{\partial z\partial x} = y\frac{\partial^2 u}{\partial z\partial y}$. [6]

ANS: $U=e^{xyz} f\left(\frac{xy}{z}\right)$

$$\frac{\partial u}{\partial x} = e^{xyz} \left[f' \left(\frac{xy}{z} \right) \times \frac{y}{z} \right] + f \left(\frac{xy}{z} \right) [e^{xyz} \times xy]$$

$$x\frac{\partial u}{\partial x} = e^{xyz} \left[\frac{xy}{z} f' \left(\frac{xy}{z} \right) + xyzf \left(\frac{xy}{z} \right) \right] \dots\dots\dots (1)$$

$$\frac{\partial u}{\partial y} = e^{xyz} \left[f' \left(\frac{xy}{z} \right) \times \frac{x}{z} \right] + f \left(\frac{xy}{z} \right) [e^{xyz} \times xz]$$

$$y\frac{\partial u}{\partial y} = e^{xyz} \left[\frac{xy}{z} f' \left(\frac{xy}{z} \right) + xyzf \left(\frac{xy}{z} \right) \right] \dots\dots\dots (2)$$

$$\frac{\partial u}{\partial z} = e^{xyz} \left[f' \left(\frac{xy}{z} \right) \times \frac{xy}{z^2} \right] + f \left(\frac{xy}{z} \right) [e^{xyz} \times xy]$$

$$z\frac{\partial u}{\partial z} = e^{xyz} \left[-\frac{xy}{z} f' \left(\frac{xy}{z} \right) + xyzf \left(\frac{xy}{z} \right) \right] \dots\dots\dots (3)$$

Adding 1 and 3, we get
 $x\frac{\partial u}{\partial x} + z\frac{\partial u}{\partial z} = 2e^{xyz} xyzf \left(\frac{xy}{z} \right) = 2xyzu$

Adding 2 and 3, we get
 $y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 2e^{xyz} xyzf \left(\frac{xy}{z} \right) = 2xyzu$

For deduction,
 $x\frac{\partial u}{\partial x} + z\frac{\partial u}{\partial z} = 2xyzu$

Diff w.r.t z

$$x\frac{\partial^2 u}{\partial x\partial z} + \left[z\frac{\partial^2 u}{\partial z^2} + \frac{\partial u}{\partial z} (1) \right] = 2xy \left[z\frac{\partial u}{\partial z} + u(1) \right]$$

$$x\frac{\partial^2 u}{\partial x\partial z} = (2xyz - 1)\frac{\partial u}{\partial z} - z\frac{\partial^2 u}{\partial z^2} + 2xyu \dots\dots\dots (4)$$

Similarly,

$$y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 2xyzu$$

Diff w.r.t z and solving, we get

$$y\frac{\partial^2 u}{\partial y\partial z} = (2xyz - 1)\frac{\partial u}{\partial z} - z\frac{\partial^2 u}{\partial z^2} + 2xyu \dots\dots\dots (5)$$

\therefore From 4 and 5, we get
 $x\frac{\partial^2 u}{\partial z\partial x} = y\frac{\partial^2 u}{\partial z\partial y}$.

(b)By using Regular falsi method solve $2x - 3\sin x - 5 = 0$. [6]

ANS: Let $f(x) = 2x - 3\sin x - 5$

- $f(1) = -5$
- $f(2) = -5.5244$
- $f(3) = -3.7379 < 0$
- $f(4) = 0.5766 > 0$

\therefore Roots lies between 2 and 3

Iteration	a	b	f(a)	f(b)	$x = \frac{af(b)-bf(a)}{f(b)-f(a)}$	f(x)
I.	2	3	-3.7279	0.5766	2.8660	-0.0841
II.	2.866	3	-0.0841	0.5766	2.8831	-0.0009
III.	2.8831	3	0.0009	0.5766	2.8833	-

(c) if $y = \sin[\log(x^2+2x+1)]$ then prove that $(x+1)^2 y_{n+2} + (2n+1)(x+1)y_{n+1} + (n^2+4)y_n = 0$. [8]

ANS: We have,

$$y = \sin[\log(x^2+2x+1)]$$

Diff with x

$$y_1 = \cos[\log(x^2+2x+1)] \times \frac{1}{x^2+2x+1} \times (2x+2)$$

$$y_1 = \cos[\log(x^2+2x+1)] \times \frac{2(x+1)}{(x+1)^2}$$

$$y_1 = \cos[\log(x^2+2x+1)] \times \frac{2}{x+1}$$

$$(x+1)y_1 = 2 \cos[\log(x^2+2x+1)]$$

Diff with x again,

$$(x+1)y_2 + y_1 = -2 \sin[\log(x^2+2x+1)] \times \frac{1}{x^2+2x+1} \times (2x+2)$$

$$(x+1)y_2 + y_1 = -2 \sin[\log(x^2+2x+1)] \times \frac{2(x+1)}{(x+1)^2}$$

$$(x+1)^2 y_2 + (x+1)y_1 = -4 \sin[\log(x^2+2x+1)]$$

$$(x+1)^2 y_2 + (x+1)y_1 = -4y$$

By Leibnitz Theorem,

$$\left[y_{n+2}(x+1)^2 + n y_{n+1} \cdot 2(x+1) + \frac{n(n-1)}{2!} y_n \cdot 2 \right] + [y_{n+1} \cdot (x+1) + n y_n(1)] = -4y_n$$

$$y_{n+2}(x+1)^2 + (2n+1)(x+1)y_{n+1} + (n^2+4)y_n = 0$$

Q.4] (a) State and prove Euler's Theorem for three variables. [6]

ANS:

Statement: If $u=f(x, y, z)$ is a homogeneous function of degree n, then -

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = nu$$

Let, $u=f(x, y, z)$ is a homogeneous function of degree n.

Putting $X = x t, Y = y t, Z = z t$.

$$f(X, Y, Z) = t^n f(x, y, z) \dots \dots \dots (1)$$

Diff LHS w.r.t t,

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$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial t}$$

$$\frac{\partial f}{\partial t} = x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + z \frac{\partial f}{\partial z} \dots\dots (2)$$

Diff RHS w.r.t. t,

$$\frac{\partial f}{\partial t} = nt^{n-1}f(x, y, z)$$

Now put t = 1, we get $\frac{\partial f}{\partial t} = nf(x, y, z) \dots\dots\dots (3)$

From equation 2 and 3, we get

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + z \frac{\partial f}{\partial z} = nf(x, y, z)$$

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + z \frac{\partial f}{\partial z} = nu$$

Hence proved

(b) By using De Moirés Theorem obtain tan 5θ in terms of tan θ and show that $1 - 10 \tan^2(\frac{\pi}{10}) + 5 \tan^4(\frac{\pi}{10}) = 0$. [6]

ANS: $(\cos 5\theta + i \sin 5\theta) = (\cos \theta + i \sin \theta)^5$
 $= \cos^5 \theta + 5 \cos^4 \theta i \sin \theta + 10 \cos^3 \theta i^2 \sin^2 \theta + 10 \cos^2 \theta i^3 \sin^3 \theta + 5 \cos \theta i^4 \sin^4 \theta + i^5 \sin^5 \theta$
 $= \cos^5 \theta + 5 \cos^4 \theta i \sin \theta - 10 \cos^3 \theta \sin^2 \theta - 10i \cos^2 \theta \sin^3 \theta + 5 \cos \theta \sin^4 \theta + i \sin^5 \theta$
 $= (\cos^5 \theta - 10 \cos^3 \theta \sin^2 \theta + 5 \cos \theta \sin^4 \theta) + i (5 \cos^4 \theta \sin \theta - 10 \cos^2 \theta \sin^3 \theta + \sin^5 \theta)$

Equating both sides we get,

$$\therefore \cos 5\theta = \cos^5 \theta - 10 \cos^3 \theta \sin^2 \theta + 5 \cos \theta \sin^4 \theta$$

$$\therefore \sin 5\theta = 5 \cos^4 \theta \sin \theta - 10 \cos^2 \theta \sin^3 \theta + \sin^5 \theta$$

But $\tan 5\theta = \sin 5\theta / \cos 5\theta$

$$= (5 \cos^4 \theta \sin \theta - 10 \cos^2 \theta \sin^3 \theta + \sin^5 \theta) / (\cos^5 \theta - 10 \cos^3 \theta \sin^2 \theta + 5 \cos \theta \sin^4 \theta)$$

Dividing by $\cos^5 \theta$

$$\tan 5\theta = \frac{5 \tan \theta - 10 \tan^3 \theta + \tan^5 \theta}{1 - 10 \tan^2 \theta + 5 \tan^4 \theta}$$

for deduction, put $\theta = \frac{\pi}{10}$

$$\cot 5 \times \frac{\pi}{10} = \frac{1 - 10 \tan^2 \frac{\pi}{10} + 5 \tan^4 \frac{\pi}{10}}{5 \tan \frac{\pi}{10} - 10 \tan^3 \frac{\pi}{10} + \tan^5 \frac{\pi}{10}}$$

$$\therefore 1 - 10 \tan^2(\frac{\pi}{10}) + 5 \tan^4(\frac{\pi}{10}) = 0.$$

Hence proved.

(c) Investigate for what values of λ and μ the equations [8]

$$2x + 3y + 5z = 9$$

$$7x + 3y - 2z = 8$$

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$2x + 3y + \lambda z = \mu$ have

- A. No solutions
- B. Unique solutions
- C. An infinite number of solutions.

ANS: Consider the system of equation of

$$2x + 3y + 5z = 9$$

$$7x + 3y - 2z = 8$$

$$2x + 3y + \lambda z = \mu$$

The above system is given as $Ax=B$

$$\begin{bmatrix} 7 & 3 & -2 \\ 2 & 3 & 5 \\ 2 & 3 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 9 \\ \mu \end{bmatrix}$$

Where $A = \begin{bmatrix} 7 & 3 & -2 \\ 2 & 3 & 5 \\ 2 & 3 & \lambda \end{bmatrix}$ $x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

And $B = \begin{bmatrix} 8 \\ 9 \\ \mu \end{bmatrix}$

$R_3 - R_2$

$$\begin{bmatrix} 7 & 3 & -2 \\ 2 & 3 & 5 \\ 0 & 0 & \lambda - 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 9 \\ \mu - 9 \end{bmatrix}$$

(A) For no solution,

$$\rho(A) \neq \rho(A B)$$

$$\therefore \lambda - 5 = 0 \text{ and } \mu - 9 \neq 0$$

$$\therefore \lambda = 5 \text{ and } \mu \neq 9$$

(B) For a unique solution

$$\rho(A) = \rho(A B) = 3$$

$$\therefore \lambda - 5 \neq 0 \text{ and } \mu \text{ may be anything}$$

$$\therefore \lambda \neq 5 \text{ for all values of } \mu$$

(C) For infinite solutions,

$$\rho(A) = \rho(A B) < 3$$

$$\therefore \lambda - 5 = 0 \text{ and } \mu - 9 = 0$$

$$\therefore \lambda = 5 \text{ and } \mu = 9$$

Q.5] (a) Find n^{th} derivative of $\frac{1}{x^2+a^2}$.

[6]

ANS: $y = \frac{1}{x^2+a^2}$

$$y = \frac{1}{(x+ai)(x-ai)}$$

Let $\frac{1}{(x+ai)(x-ai)} = \frac{A}{(x+ai)} + \frac{B}{(x-ai)}$

$$1 = A(x - ai) + B(x + ai)$$

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Put $x = ai$,

$$1 = B(2ai)$$

$$B = \frac{1}{2ai}$$

Put $x = -ai$, we get

$$A = -\frac{1}{2ai}$$

$$\therefore y = \frac{1}{(x+ai)} + \frac{1}{(x-ai)}$$

$$\therefore y = \frac{1}{2ai} \left[\frac{1}{(x+ai)} - \frac{1}{(x-ai)} \right]$$

Diff n times, we get

$$y_n = \frac{1}{2ai} \left[\frac{(-1)^n n!}{(x-ai)^{n+1}} - \frac{(-1)^n n!}{(x+ai)^{n+1}} \right].$$

(b) If $z = f(x, y)$ where $x = e^u + e^{-v}$, $y = e^{-u} - e^v$ then prove that $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x \frac{\partial z}{\partial x} - y \frac{\partial z}{\partial y}$.

[6]

ANS: Given: $z = f(x, y)$, $x = e^u + e^{-v}$ (1)

$$y = e^{-u} - e^v \text{ (2)}$$

By Chain Rule,

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u} \text{ (3)}$$

And

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v} \text{ (4)}$$

\therefore From equation 1 and 2,

$$\begin{aligned} \frac{\partial x}{\partial u} &= e^u & \frac{\partial x}{\partial v} &= -e^{-v} \\ \frac{\partial y}{\partial u} &= -e^{-u} & \frac{\partial y}{\partial v} &= -e^v \end{aligned}$$

\therefore From equation 3 and 4,

$$\frac{\partial z}{\partial u} = e^u \frac{\partial z}{\partial x} - e^{-u} \frac{\partial z}{\partial y} \text{ (5)}$$

And

$$\frac{\partial z}{\partial v} = -e^{-v} \frac{\partial z}{\partial x} - e^v \frac{\partial z}{\partial y} \text{ (6)}$$

By Subtracting Equation 5 and 6,

$$\begin{aligned} \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} &= (e^u + e^{-v}) \frac{\partial z}{\partial x} - (e^{-u} - e^v) \frac{\partial z}{\partial y} \\ &= x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} \text{ (By using equation 1 and 2)} \end{aligned}$$

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x \frac{\partial z}{\partial x} - y \frac{\partial z}{\partial y}$$

Hence proved.

(c) Solve using Gauss Jacobi Iteration method

[8]

$$2x + 12y + z - 4w = 13$$

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$$13x + 5y - 3z + w = 18$$

$$2x + y - 3z + 9w = 31$$

$$3x - 4y + 10z + w = 29$$

ANS:

$$x = \frac{18-5y+3z-w}{13}$$

$$y = \frac{13-2x-z+4w}{12}$$

$$z = \frac{29-3x+4y-w}{10}$$

$$w = \frac{31-2x-y+3z}{9}$$

Iteration	x	y	z	w
1	1.3846	1.0833	2.9	3.4444
2	1.3722	1.7590	2.5735	3.9831
3	0.9956	1.9679	2.7936	3.8019
4	0.9800	1.9519	3.0083	3.9357
5	1.0254	1.9812	2.9932	4.0126
6	1.0047	2.0005	2.9836	3.9942
7	0.9965	1.9987	2.9994	3.9934
8	1.0009	1.9984	3.0012	4.0007
9	1.0008	2.0000	2.9990	4.0004
10	0.9997	2.0001	2.9997	3.9995
11	0.9999	1.9999	3.0002	4.0000
12	1.0001	2	3	4.0001
13	1	2	3	4

$\therefore x = 1, y = 2, z = 3, w = 4.$

Q.6] (a) If $y = \log \left[\tan \left(\frac{\pi}{4} + \frac{x}{2} \right) \right]$ Prove that

[6]

I. $\tan h \frac{y}{2} = \tan \frac{x}{2}$

II. $\cos hy \cos x = 1$

ANS: 1] $\sin h \frac{y}{2} = \frac{e^{\frac{y}{2}} - e^{-\frac{y}{2}}}{2}$

$$\cos h \frac{y}{2} = \frac{e^{\frac{y}{2}} + e^{-\frac{y}{2}}}{2}$$

$$\tan h \frac{y}{2} = \frac{\sin h \frac{y}{2}}{\cos h \frac{y}{2}}$$

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$$\begin{aligned} & \frac{\frac{y}{e^2} - \frac{-y}{2}}{e^2 - e^{\frac{-y}{2}}} \\ &= \frac{2}{\frac{y}{e^2} + \frac{-y}{2}} \\ &= \frac{e^y - 1}{e^y + 1} \end{aligned}$$

$$\text{But } e^u = \frac{1 + \tan \frac{x}{2}}{1 - \tan \frac{x}{2}}$$

$$\begin{aligned} \therefore \tanh \frac{y}{2} &= \frac{\frac{1 + \tan \frac{x}{2}}{1 - \tan \frac{x}{2}} - 1}{\frac{1 + \tan \frac{x}{2}}{1 - \tan \frac{x}{2}} + 1} \\ &= \frac{1 + \tan \frac{x}{2} - 1 + \tan \frac{x}{2}}{1 + \tan \frac{x}{2} + 1 - \tan \frac{x}{2}} \end{aligned}$$

$$= \tan \frac{x}{2}$$

$$\therefore \tanh \frac{y}{2} = \tan \frac{x}{2}$$

$$2] y = \log \left[\tan \left(\frac{\pi}{4} + \frac{x}{2} \right) \right]$$

$$\begin{aligned} e^y &= \tan \left(\frac{\pi}{4} + \frac{x}{2} \right) \\ &= \frac{\tan \frac{\pi}{4} + \tan \frac{x}{2}}{1 - \tan \frac{\pi}{4} \tan \frac{x}{2}} \end{aligned}$$

$$e^y = \frac{1 + \tan \frac{x}{2}}{1 - \tan \frac{x}{2}}$$

$$e^{-y} = \frac{1 - \tan \frac{x}{2}}{1 + \tan \frac{x}{2}}$$

$$\begin{aligned} \cosh y &= \frac{e^y + e^{-y}}{2} \\ &= \frac{\frac{1 + \tan \frac{x}{2}}{1 - \tan \frac{x}{2}} + \frac{1 - \tan \frac{x}{2}}{1 + \tan \frac{x}{2}}}{2} \\ &= \frac{(1 + \tan \frac{x}{2})^2 + (1 - \tan \frac{x}{2})^2}{2(1 - \tan^2 \frac{x}{2})} \\ &= \frac{1 + 2 \tan \frac{x}{2} + \tan^2 \frac{x}{2} + 1 + \tan^2 \frac{x}{2} - 2 \tan \frac{x}{2}}{2(1 - \tan^2 \frac{x}{2})} \\ &= \frac{1 + \tan^2 \frac{x}{2}}{1 - \tan^2 \frac{x}{2}} \end{aligned}$$

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$$\therefore \cos hy = \frac{1}{\cos x}$$

$$\therefore \cos hy \cos x = \frac{1}{\cos x} \cdot \cos x$$

$$\cos hy \cos x = 1$$

Hence proved

(b) If $u = \sin^{-1} \left[\frac{x^{\frac{1}{3}} + y^{\frac{1}{3}}}{x^2 + y^2} \right]^{\frac{1}{2}}$ prove that

$$x^2 \frac{\partial^2 u}{\partial^2 x} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial^2 y} = \frac{\tan u}{144} [\tan^2 u + 13]. \quad [6]$$

ANS:

$$\text{Given } u = \sin^{-1} \left[\frac{x^{\frac{1}{3}} + y^{\frac{1}{3}}}{x^2 + y^2} \right]^{\frac{1}{2}}$$

$$z = \sin u = \left[\frac{x^{\frac{1}{3}} + y^{\frac{1}{3}}}{x^2 + y^2} \right]^{\frac{1}{2}} \text{ is homogeneous function in } x \text{ and } y \text{ with degree } -\frac{1}{12}$$

\therefore We have the result,

If $z = f(u)$ is homogeneous function of degree x and y then

$$x^2 \frac{\partial^2 u}{\partial^2 x} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial^2 y} = g(u) [g'(u) - 1] \text{ where } g(u) = n \frac{f(u)}{f'(u)}.$$

$$n = -\frac{1}{12}, f(u) = \sin u, f'(u) = \cos u$$

$$\therefore g(u) = -\frac{1}{12} \frac{\sin u}{\cos u}$$

$$\therefore g(u) = -\frac{1}{12} \tan u$$

$$\therefore g'(u) = -\frac{1}{12} \sec^2 u$$

By above result,

$$\begin{aligned} x^2 \frac{\partial^2 u}{\partial^2 x} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial^2 y} &= -\frac{1}{12} \left[-\frac{1}{12} \sec^2 u - 1 \right] \\ &= \frac{1}{12} \left[\frac{1}{12} \sec^2 u + 1 \right] = \frac{1}{12} \left[\frac{1 + \tan^2 u}{12} + 1 \right] \\ &= \frac{1}{144} \tan u [\tan^2 u + 13] \end{aligned}$$

$$\therefore x^2 \frac{\partial^2 u}{\partial^2 x} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial^2 y} = \frac{\tan u}{144} [\tan^2 u + 13].$$

Hence proved

(c) Expand $2x^3 + 7x^2 + x - 6$ in power of $(x - 2)$ by using Taylors Theorem.

[4]

ANS: By Taylor's series,

$$f(x) = f(a) + (x+a)f'(a) + \frac{(x+a)^2}{2!} f''(a) + \frac{(x+a)^3}{3!} f'''(a) + \dots$$

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Here,

$$f(x) = 2x^3 + 7x^2 + x - 6$$

$$f(2) = 2(2)^3 + 7(2)^2 + 2 - 6 = 40$$

$$f'(x) = 6x^2 + 14x + 1$$

$$f'(2) = 6(2)^2 + 14(2) + 1 = 53$$

$$f''(x) = 12x + 14$$

$$f''(2) = 12(2) + 14 = 38$$

$$f'''(x) = 12$$

$$f'''(2) = 12$$

$$f''''(x) = 0.$$

$$f(x) = f(2) + (x - 2)f'(2) + \frac{(x - 2)^2}{2!} f''(2) + \frac{(x - 2)^3}{3!} f'''(2) + 0$$

$$2x^3 + 7x^2 + x - 6 = 40 + (x - 2)(53) + \frac{(x - 2)^2}{2!} (38) + \frac{(x - 2)^3}{3!} (12)$$

$$2x^3 + 7x^2 + x - 6 = 2(x - 2)^3 + 19(x - 2)^2 + 53(x - 2) + 40$$

(d) Expand sec x by McLaurin's theorem considering up to x^4 term.

[4]

ANS: Let $y = \sec x$

$$y = 1 / (\cos x)$$

$$y = \frac{1}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots}$$

$$y = (1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots)^{-1}$$

$$y = 1 - (\frac{-x^2}{2} + \frac{x^4}{24}) + (\frac{-x^2}{2} + \dots)^2 + \dots$$

$$y = 1 + \frac{x^2}{2} - \frac{x^4}{24} + \frac{x^4}{4} + \dots$$

$$\therefore y = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \dots$$

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